

Fractals for Kernelization Lower Bounds, With an Application to Length-Bounded Cut Problems

Till Fluschnik^{*1}, Danny Hermelin^{†2}, André Nichterlein^{‡1}, and Rolf Niedermeier¹

¹Institut für Softwaretechnik und Theoretische Informatik, TU Berlin, Germany, {till.fluschnik, andre.nichterlein, rolf.niedermeier}@tu-berlin.de

²Department of Industrial Engineering and Management, Ben-Gurion University of the Negev, Israel, hermelin@bgu.ac.il

April 28, 2016

Abstract

Bodlaender et al.’s [SIDMA 2014] cross-composition technique is a popular method for excluding polynomial-size problem kernels for NP-hard parameterized problems. We present a new technique exploiting triangle-based fractal structures for extending the range of applicability of cross-compositions. Our technique makes it possible to prove new no-polynomial-kernel results for a number of problems dealing with length-bounded cuts. Roughly speaking, our new technique combines the advantages of serial and parallel composition. In particular, answering an open question of Golovach and Thilikos [Discrete Optim. 2011], we show that, unless $\text{NP} \subseteq \text{coNP} / \text{poly}$, the NP-hard LENGTH-BOUNDED EDGE-CUT problem (delete at most k edges such that the resulting graph has no s - t path of length shorter than ℓ) parameterized by the combination of k and ℓ has no polynomial-size problem kernel. Our framework applies to planar as well as directed variants of the basic problems and also applies to both edge and vertex deletion problems.

Key words: Fixed-parameter tractability; polynomial kernels; kernelization; kernel lower bounds; cross-compositions; graph modification problems; fractals.

1 Introduction

Lower bounds are of central concern all over computational complexity analysis. With respect to fixed-parameter tractable problems [12, 15, 18, 32], currently there are two main streams in this context: (i) ETH-based lower bounds for the running times of exact algorithms [27]¹ and (ii) lower bounds on problem kernel sizes; more specifically, the exclusion of polynomial-size problem kernels [26]. Both these research directions for lower bounds rely on plausible complexity-theoretic assumptions, namely the Exponential-Time Hypothesis (ETH) and $\text{NP} \not\subseteq \text{coNP} / \text{poly}$, respectively. In this work, we contribute to the second research direction, developing a new technique that exploits a triangle-based fractal structure in order to exclude polynomial-size problem kernels (polynomial kernels for short) for edge and vertex deletion problems in the context of length-bounded cuts.

Kernelization is a key method for designing fixed-parameter algorithms [22, 26]; among all techniques of parameterized algorithm design, it has the presumably greatest potential for delivering

^{*}Till Fluschnik acknowledges support by the DFG, project DAMM (NI 369/13-2).

[†]Danny Hermelin was supported by a DFG Mercator fellowship within the project DAMM (NI 369/13-2) while staying at TU Berlin (August 2015). He has also received funding from the People Programme (Marie Curie Actions) of the European Union’s Seventh Framework Programme (FP7/2007-2013) under REA grant agreement number 631163.11, and by the ISRAEL SCIENCE FOUNDATION (grant No. 551145/14).

[‡]From February 2016 to January 2017 on postdoctoral leave to Durham University (GB), funded by DAAD.

¹Also see the November 2015 workshop on “Satisfiability Lower Bounds and Tight Results for Parameterized and Exponential-Time Algorithms” at the Simons Institute for the Theory of Computing, <https://simons.berkeley.edu/workshops/schedule/1819>.

practically relevant algorithms. Hence, it is of key interest to explore its power and its limitations. In a nutshell, the fundamental idea of kernelization is as follows. Given a parameterized problem instance I with parameter k , in polynomial time preprocess I by applying data reduction rules in order to simplify it and reduce it to an “equivalent” instance (so-called (problem) kernel) of the same problem. For NP-hard problems the best one can hope for is a problem kernel of size polynomial in the parameter k . In a way, one may interpret kernelization (requested to run in polynomial time) as an “exact counterpart” of polynomial-time approximation algorithms. Indeed, linear-size problem kernels often imply constant-factor approximation algorithms [29, page 15]. Approximation algorithmics has a highly developed theory (having produced concepts such as MaxSNP-hardness and the famous PCP theory) for proving (relative to some plausible complexity-theoretic assumption) lower bounds on the approximation factors [1, 36, 38].

It is fair to say that in the younger field of kernelization the arsenal for proving lower bounds (particularly excluding polynomial kernels) so far is of smaller scope and needs further development. The most influential result in this context is due to Bodlaender et al. [7] and Fortnow and Santhanam [19]: Based on the assumption $\text{NP} \not\subseteq \text{coNP}/\text{poly}$, it is shown that e.g. the NP-hard graph problem LONGEST PATH parameterized by solution size has no polynomial kernel. The core tool for showing this are so-called “OR-compositions”. (Meanwhile it is known that “AND-compositions” can be used as well [16]). To ease the use of this kernel-lower-bound framework, one natural idea is to use “polynomial parameter transformations”, that is, a form of “parameter-preserving reductions” [9, 14]. An easier-to-use generalization of the OR-composition technique is given by so-called OR-cross-compositions [10]. Currently, these two approaches constitute the known core tools to exclude polynomial kernels. Building on OR-cross-compositions, we add a further tool (which we baptized “fractalism”) in order to extend the range of problems to be addressed by OR-cross-compositions. The usefulness of our new technique is substantiated by resolving an open problem posed by Golovach and Thilikos [21], here specifically applying our technique to the NP-hard LENGTH-BOUNDED EDGE-CUT problem.

Next, we discuss in some more detail OR-(cross)-compositions. Roughly speaking, the idea behind an OR-composition for a parameterized problem is to encode the logical “or” of t instances with parameter value k into a single instance of the same problem with parameter value $k' = k^{O(1)}$. In particular, given t instances, the obtained instance is a yes-instance if and only if at least one of the given instances is a yes-instance. If an OR-composition is possible, then this excludes polynomial kernels. Whereas in OR-compositions one combines instances of an NP-hard parameterized problem into one instance of a parameterized problem, in OR-cross-compositions one combines instances of classical NP-hard problems into one instance of a parameterized problem (see Section 2 for details and formal definitions).

While for some problems, for example LONGEST PATH with parameter solution size [7], a simple disjoint union yields the desired OR-composition, other problems seem to require involved constructions, for example SET COVER with parameter universe size [14]. Indeed, devising a cross-composition can be quite challenging and the task becomes even harder when considering several, seemingly orthogonal parameterizations at once. To illustrate the problem with such combined parameters, let us consider the problem LENGTH-BOUNDED EDGE-CUT (LBEC). Herein, an undirected graph $G = (V, E)$ with $s, t \in V$, and two integers $k, \ell \in \mathbb{N}$ are given, and the question is whether it is possible to delete at most k edges such that the shortest s - t path is of length at least ℓ . Using a simple branching algorithm, one can show that $\text{LBEC}(k, \ell)$ is fixed-parameter tractable for the combined parameter (k, ℓ) [4, 21]. To exclude the existence of a polynomial kernel for $\text{LBEC}(k, \ell)$, we would like to apply the OR-cross-composition framework to the problem, and as a natural candidate for the input problem we decide for LBEC itself.

A standard approach to applying the OR-cross-composition to a problem like LBEC would be to concatenate the input instances on the source and sink vertices, also referred to as “serial” composition. To this end, one needs some additional gadgets to ensure that only in one instance edges are deleted. This form of composition, however, induces a dependency of the second parameter ℓ on the number of instances, which is not allowed. Another standard approach is introducing a “global” sink and source vertex, and connecting all source vertices with the global source and all sink vertices, also referred to as a “parallel” composition. This form of composition would keep ℓ small enough, but induces a dependency of the first parameter k on the number of instances. Summarizing, the parameter k seems to ask for a serial composition and the parameter ℓ seems to ask

Problem	edge deletion			
	directed		undirected	
	planar	general	planar	general
LBEC(k, ℓ)	No PK [Thm. 4]	No PK [Thm. 2]	No PK [Thm. 4]	No PK [Thm. 1]
MDED(k, ℓ)	No PK [Thm. 8]	No PK [Thm. 8]	No PK [Thm. 7]	No PK [Thm. 6]
DSCT(k, ℓ)	No PK [Thm. 11]	No PK [Thm. 10]	PK [39]	?

Table 1: Survey of the concrete results of this paper (under the assumption that $\text{NP} \not\subseteq \text{coNP} / \text{poly}$). PK stands for polynomial kernel and a “?” indicates that it is open whether a polynomial kernel exists. We remark that the no-polynomial-kernel results for LBEC(k, ℓ) on directed graphs still hold for directed acyclic graphs. Note that we claim without proof that, except for the planar variants, our proofs also transfer to the vertex deletion case, both for directed and undirected graphs.

for a parallel composition. For some problems using a tree as “instance selector” was helpful, see for example Bevern et al. [6] or Bazgan et al. [3]. The problem with trees is that they introduce small (constant-size) s - t cuts, which is problematic for LENGTH-BOUNDED EDGE-CUT. In this work, we introduce a fractal structure as instance selector which has the nice properties of trees but does not introduce small cuts. So, our fractal structure helps to exclude polynomial kernels for several problems.

Our contributions. Our main technical contribution is to introduce a family of graphs that we call T-fractals and that build on triangles. T-fractals feature a fractal-like structure, in the sense of self-similarity and scale-invariance. Using these T-fractals in OR-cross-compositions, we show that the following parameterized graph modification problems and several of their variants do not admit polynomial kernels (unless $\text{NP} \subseteq \text{coNP} / \text{poly}$):

- LENGTH-BOUNDED EDGE-CUT(k, ℓ) (LBEC(k, ℓ)), where k is the number of edges to delete, and ℓ is the lower bound on the length of the shortest path.
- MINIMUM DIAMETER EDGE DELETION(k, ℓ) (MDED(k, ℓ)), where k is the number of edges to delete, and ℓ is the lower bound on the target diameter.
- DIRECTED SMALL CYCLE TRANSVERSAL(k, ℓ) (DSCT(k, ℓ)), where k is the number of edges to delete, and ℓ is the lower bound on the shortest induced cycle.

Table 1 surveys our no-polynomial-kernel results and spots an open question.

The graph modification problems LENGTH-BOUNDED EDGE-CUT (LBEC), MINIMUM DIAMETER EDGE DELETION (MDED), and DIRECTED SMALL CYCLE TRANSVERSAL (DSCT) are defined as follows. The LBEC problem asks, given an undirected graph $G = (V, E)$, two vertices $s, t \in V$, and two integers k, ℓ , whether there are at most k edge deletions such that the shortest s - t path is of length at least ℓ . The MDED problem asks, given an undirected connected graph $G = (V, E)$ and two integers k, ℓ , whether there are at most k edge deletions such that the remaining graph remains connected and has diameter at least ℓ . The DSCT problem asks, given a directed graph $G = (V, E)$ and two integers k, ℓ , whether there are at most k edge deletions such that the remaining graph has no cycle of length smaller than ℓ . In addition, we consider several variants (planar, directed, vertex deletion) of these problems.

2 Preliminaries

We use standard notation from parameterized complexity [12, 15, 18, 32] and graph theory [13, 37].

Graph Theory Let $G = (V, E)$ be a graph. We denote by $V(G)$ the vertex set of G and by $E(G)$ the edge set of G . For a vertex set $W \subseteq V(G)$ (edge set $F \subseteq E(G)$), we denote by $G[W]$ ($G[F]$) the subgraph of G induced by the vertex set W (edge set F). For $C \subseteq V(G)$ ($C \subseteq E(G)$)

we write $G - C$ for the graph G where all vertices (edges) in C are deleted. Note that the deletion of a vertex implies the deletion of all its incident edges.

Let G be an undirected, connected graph. An edge cut $C \subseteq E(G)$ is a set of edges such that the graph $G - C$ is not connected. If G is a directed, connected graph, then $C \subseteq E(G)$ is an edge cut if C is an edge cut in the undirected version of G . Let $s, t \in V(G)$ be two vertices in G . An s - t edge cut C is an edge cut such that the vertices s and t are not connected in $G - C$. An s - t edge cut C is called *minimal* if $C \setminus \{e\}$ is not an s - t edge cut in G for all $e \in C$. An s - t edge cut C is called *minimum* if there is no s - t edge cut C' in G such that $|C'| < |C|$.

A path is a connected graph with exactly two vertices of degree one and no vertex of degree at least three. We call the vertices with degree one the endpoints of the path. An s - t path is a path where the vertices s and t are the endpoints of the path. The length of a path is the number of edges in the path. In directed graphs, an s - t path is a path where all arcs are directed toward t . The *diameter* of a graph G is the maximum length of any shortest v - w path over all $v, w \in V(G)$, $v \neq w$.

A cycle is a connected graph where every vertex has degree exactly two. The length of a cycle is the number of edges in the cycle. In directed graphs, a cycle is a connected graph where every vertex has outdegree and indegree exactly one. The *girth* of a graph G is the length of the shortest cycle (contained) in graph G .

Given two graphs $G = (V, E)$ and $G' = (V', E')$, we say that G' is the graph obtained from G by *splitting* a vertex $v \in V(G)$ into k vertices $v_1, \dots, v_k \in V(G')$, called the *split vertices*, if the following holds: (i) $V' = (V \setminus \{v\}) \cup \{v_1, \dots, v_k\}$, (ii) $G[V \setminus \{v\}]$ is isomorphic to $G'[V' \setminus \{v_1, \dots, v_k\}]$, and (iii) $\{v, w\} \in E \Rightarrow$ it exists exactly one $i \in [k]$ with $\{v_i, w\} \in E'$.

Given a graph $G = (V, E)$ and two non-adjacent vertices $v, w \in V$, we say we *merge* the vertices v and w if we add a new vertex vw to V as well as the edge set $\{\{vw, x\} \mid \{x, v\} \in E\} \cup \{\{vw, x\} \mid \{x, w\} \in E\}$ to E , and we delete the vertices v and w and all incident edges to v and w .

Parameterized Complexity A *parameterized problem* is a set of instances (\mathcal{I}, k) where $\mathcal{I} \in \Sigma^*$ for a finite alphabet Σ , and $k \in \mathbb{N}$ is the *parameter*. A parameterized problem L is *fixed-parameter tractable (fpt)* if it can be decided in $f(k) \cdot |\mathcal{I}|^{O(1)}$ time whether $(\mathcal{I}, k) \in L$, where f is a computable function only depending on k . We say that two instances (\mathcal{I}, k) and (\mathcal{I}', k') of parameterized problems P and P' are *equivalent* if (\mathcal{I}, k) is YES for P if and only if (\mathcal{I}', k') is YES for P' . A *kernelization* is an algorithm that, given an instance (\mathcal{I}, k) of a parameterized problem P , computes in polynomial time an equivalent instance (\mathcal{I}', k') of P (the *kernel*) such that $|\mathcal{I}'| + k' \leq f(k)$ for some computable function f only depending on k . We say that f measures the *size* of the kernel, and if $f \in k^{O(1)}$, we say that P admits a polynomial kernel. We remark that a decidable parameterized problem is fixed-parameter tractable if and only if it admits a kernel [11].

Given an NP-hard problem L , an equivalence relation \mathcal{R} on the instance of L is a *polynomial equivalence relation* if (i) one can decide for any two instances in time polynomial in their sizes whether they belong to the same equivalence class, and (ii) for any finite set S of instances, \mathcal{R} partitions the set into at most $(\max_{x \in S} |x|)^{O(1)}$ equivalence classes.

Definition 1. Given an NP-hard problem L , a parameterized problem P , and a polynomial equivalence relation \mathcal{R} on the instances of L , an OR-cross-composition of L into P (with respect to \mathcal{R}) is an algorithm that takes ℓ \mathcal{R} -equivalent instances $\mathcal{I}_1, \dots, \mathcal{I}_\ell$ of L and constructs in time polynomial in $\sum_{i=1}^{\ell} |\mathcal{I}_i|$ an instance (\mathcal{I}, k) such that

1. k is polynomially upper-bounded in $\max_{1 \leq i \leq \ell} |\mathcal{I}_i| + \log(\ell)$ and
2. (\mathcal{I}, k) is YES for P if and only if there is at least one $\ell' \in [\ell]$ such that $\mathcal{I}_{\ell'}$ is YES for L .

If a parameterized problem P admits an OR-cross-composition for some NP-hard problem L , then P does not admit a polynomial kernel with respect to its parameterization, unless $\text{NP} \subseteq \text{coNP/poly}$ [10]. We remark that we can assume that $\ell = 2^j$ for some $j \in \mathbb{N}$ since we can add trivial NO-instances from the same equivalence class to reach a power of two. We refer to the survey of Kratsch [26] for an overview on kernelization and lower bounds.

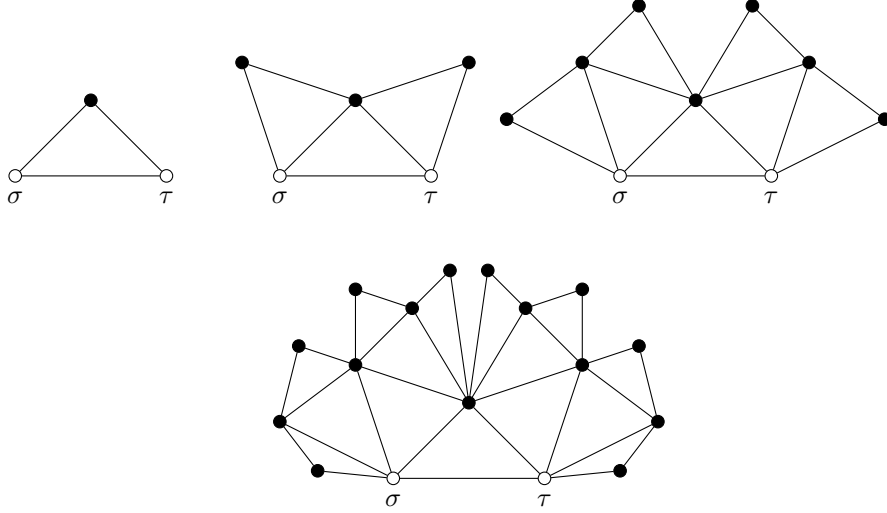


Figure 1: T-fractals $\Delta_1, \Delta_2, \Delta_3, \Delta_4$. The two special vertices σ and τ are highlighted by empty circles.

3 The “Fractalism” Technique

In this section, we describe our new technique based on *triangle fractals* (*T-fractals* for short). We provide a general construction scheme for cross-compositions using T-fractals. To this end, we first define T-fractals and then discuss several of their properties in Section 3.1. Furthermore, we present in Section 3.2 a directed variant and provide two “construction manuals” for an application of T-fractals in cross-compositions in Section 3.3.

Roughly speaking, a T-fractal can be constructed by iteratively putting triangles on top of each other, see Fig. 1 for four examples.

Definition 2. For $q \geq 1$, the q -T-fractal Δ_q is the graph constructed as follows:

- (1) Set $\Delta_0 := \{\sigma, \tau\}$ with $\{\sigma, \tau\}$ being a “marked edge” with endpoints σ and τ , subsequently referred to as special vertices.
- (2) Let F be the set of marked edges.
- (3) For each edge $e \in F$, add a new vertex and connect it by two new edges with the endpoints of e , and mark the two added edges.
- (4) Unmark all edges in F .
- (5) Repeat (2)-(4) $q - 1$ times.

The fractal structure of Δ_q might be easier to see when considering the following equivalent recursive definition of Δ_q : For the base case we define $\Delta_0 := \{\sigma, \tau\}$ as in Definition 2. Then, the q -T-fractal Δ_q is constructed as follows. Take two $(q - 1)$ -T-fractals Δ'_{q-1} and Δ''_{q-1} , where σ', τ' and σ'', τ'' are the special vertices of Δ'_{q-1} and Δ''_{q-1} , respectively. Then Δ_q is obtained by merging the vertices τ' and σ'' and adding the edge $\{\sigma', \tau''\}$. Set $\sigma = \sigma'$ and $\tau = \tau''$ as the special vertices of Δ_q . We remark that we make use of the recursive structure in later proofs.

In the i th execution of (2)-(4) in Definition 2, we obtain 2^{i-1} many triangles. We say that these triangles have depth i . The *boundary* $B_i \subseteq E(\Delta_q)$, $i \in [q]$, are those edges of the triangles of depth i which are not edges of the triangles of depth $i - 1$. As a convention, the edge $\{\sigma, \tau\}$ connecting the two special vertices σ and τ forms the boundary B_0 . We refer to Fig. 2 for an illustration of the boundaries in the T-fractal Δ_4 . Moreover, by construction, we obtain the following:

Observation 1. In every T-fractal, each boundary forms a σ - τ path, and all boundaries are pairwise edge-disjoint.

Note that the boundary B_q contains $p = 2^q$ edges. Thus, the number of edges in Δ_q is $\sum_{i=0}^q 2^i = 2^{q+1} - 1 = 2 \cdot p - 1$. Further observe that all vertices of Δ_q are incident with the edges in B_q , and B_q forms a σ - τ path. Hence, Δ_q contains $p + 1$ vertices.

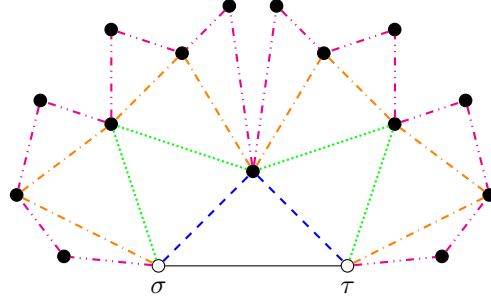


Figure 2: Highlighting the different boundaries of Δ_4 by line-types (solid: boundary B_0 ; dashed: boundary B_1 ; dotted: boundary B_2 ; dash-dotted: boundary B_3 ; dash-dot-dotted: boundary B_4).

Reducing the Weighted to the Unweighted Case. In the remainder of the paper, we focus on the unweighted case of T-fractals without multiple edges or loops. This is possible due to the following reduction of the weighted to the unweighted case. Equip the T-fractal with an *edge cost*, that is, the cost for deleting any edge in the T-fractal. If $c \in \mathbb{N}$ is the edge cost of Δ_q , then we write Δ_q^c (we drop the superscript if $c = 1$). To reduce to the case with an unweighted, simple graph, we add $c - 1$ further copies for each edge. Thus, to make two adjacent vertices non-adjacent, it requires c edge-deletions. To make the graph simple, we subdivide each edge. We remark that in this way we double the distances of the vertices in the original T-fractal. Thus, whenever we consider distances in the fractal with edge cost and the graph obtained by the reduction above, we have to take into account a factor of two.

3.1 Properties of T-Fractals

The goal of this subsection is to prove several properties of T-fractals that are used in later constructions. Some key properties of T-fractals appear in the context of σ - τ edge cuts in Δ_q . To prove other properties, we later introduce the notion of the dual structure behind the T-fractals.

The minimum edge cuts in Δ_q will play a central role when using T-fractals in cross-compositions since the minimum edge cuts serve as instance selectors (see Section 3.3). First, we discuss the size and structure of the minimum edge cuts in Δ_q .

Lemma 1. *Every minimum σ - τ edge cut in Δ_q is of size $q + 1$.*

Proof. Let C be a minimum σ - τ edge cut in Δ_q . Note that the degrees of σ and τ are exactly $q + 1$, and thus $|C| \leq q + 1$. Moreover, the boundaries in Δ_q are pairwise edge-disjoint and each boundary forms a σ - τ path (Observation 1). Since Δ_q contains $q + 1$ boundaries, it follows that there are at least $q + 1$ disjoint σ - τ paths in Δ_q . Menger's theorem [30] states that in a graph with distinct source and sink, the maximum number of disjoint source-sink paths equals the minimum size of any source-sink edge cut. Thus, by Menger's theorem, it follows that $|C| \geq q + 1$. Hence $|C| = q + 1$. \square

From the fact that the boundaries are pairwise edge-disjoint and each boundary forms a σ - τ path, we can immediately derive the following from Lemma 1.

Corollary 1. *Every minimum σ - τ edge cut in Δ_q contains exactly one edge of each boundary.*

In the following we describe a (hidden) dual structure in Δ_q , that is, a complete binary tree with p leaves. We refer to Fig. 3 for an example of the dual structure in Δ_3 . To talk about the dual structure by means of duality of plane graphs, we need a plane embedding of Δ_q . Hence we assume that Δ_q is embedded as in Fig. 1 (iteratively extended). By T_q we denote the dual structure in Δ_q , where the vertex dual to the outer face is split into $p + 1$ vertices such that each split vertex has degree exactly one. We consider the split vertex incident with the vertex dual to the triangle containing the edge $\{\sigma, \tau\}$ as the root vertex of the dual structure T_q . Thus, the other split vertices correspond to the leaves of the dual structure T_q . Note that the depth of a triangle one-to-one corresponds to the depth of the dual vertex in T_q .

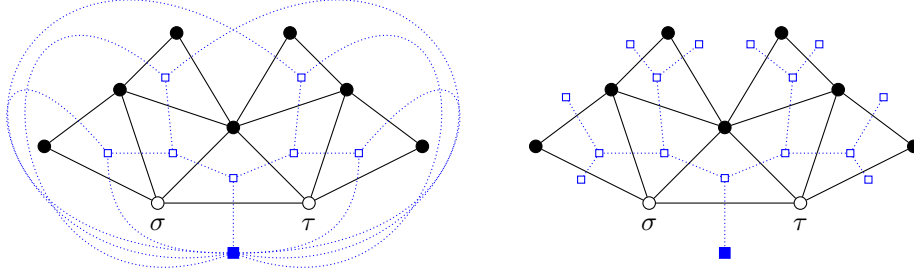


Figure 3: Left: The T-fractal Δ_3 (circles and solid lines) and its dual graph (squares and dotted lines). The filled square is the vertex dual to the outer face in the dual graph. Right: The T-fractal Δ_3 (circles and solid lines) and its dual structure T_3 , illustrated by squares and dotted lines, where the filled square corresponds to the root of the dual structure.

Observe that there is a one-to-one correspondence between the edges in T_q and the edges in Δ_q . The following lemma states duality of root-leaf paths in T_q and minimum σ - τ edge cuts in Δ_q , demonstrating the utility of the dual structure T_q .

Lemma 2. *There is a one-to-one correspondence between root-leaf paths in the dual structure T_q of Δ_q and minimum σ - τ edge cuts in Δ_q . Moreover, there are exactly $p = 2^q$ pairwise different minimum σ - τ edge cuts in Δ_q .*

Proof. Observe that each path from the root to a leaf in the dual structure T_q corresponds to a cycle in the dual graph. It is well-known that there is a one-to-one correspondence between minimal edge cuts in a plane graph and cycles in its dual graph [13, Proposition 4.6.1]. Herein, every cycle in the dual graph that “cuts” the edge $\{\sigma, \tau\}$ in Δ_q is a root-leaf path in T_q . Thus, the only minimal σ - τ edge cuts are those corresponding to the root-leaf paths. By the one-to-one correspondence of the depth of the triangles in Δ_q and the depth of the vertices in T_q , these edge cuts are of cardinality $q + 1$. Hence, by Lemma 1, these edge cuts are minimum edge cuts.

Since $|B_q| = p$, there are exactly p leaves in T_q , and thus there are exactly p different root-leaf paths in T_q . It follows that the number of pairwise different minimum σ - τ edge cuts in Δ_q is exactly $p = 2^q$. \square

Further, we obtain the following.

Lemma 3. *Let C be a minimum σ - τ edge cut in Δ_q . Let $\{x, y\} = C \cap B_q$, where x is in the same connected component as σ in $\Delta_q - C$. Then $\text{dist}(\sigma, x) + \text{dist}(y, \tau) = q$ in $\Delta_q - C$.*

Proof. We prove the lemma by induction on q . For the base case $q = 0$, observe that $C = \{\sigma, \tau\}$ and $\text{dist}_{\Delta_0 - C}(\sigma, x) + \text{dist}_{\Delta_0 - C}(y, \tau) = 0$.

For the induction step, assume that the statement of the lemma is true for Δ_{q-1} . Now, let C be a minimum σ - τ edge cut in Δ_q . Hence, $\{\sigma, \tau\} \in C$. Denote by u the (unique) vertex that is adjacent to the two special vertices σ and τ . Let Δ'_{q-1} and Δ''_{q-1} be the two $(q-1)$ -T-subfractals of Δ_q , so that Δ'_{q-1} (Δ''_{q-1}) has the special vertices σ and u (u and τ). By Lemma 2, the minimum σ - τ edge cut C corresponds to a root-leaf path in T_q . Hence, $C' := C \setminus \{\sigma, \tau\}$ is either a subset of $E(\Delta'_{q-1})$ or of $E(\Delta''_{q-1})$. Assume w.l.o.g. that $C' \subseteq E(\Delta'_{q-1})$. It follows from the induction hypothesis that $\text{dist}_{\Delta'_{q-1} - C'}(\sigma, x) + \text{dist}_{\Delta'_{q-1} - C'}(y, u) = q - 1$. Since $\text{dist}_{\Delta_q - C}(y, \tau) = \text{dist}_{\Delta'_{q-1} - C'}(y, u) + 1$, it follows that $\text{dist}_{\Delta_q - C}(\sigma, x) + \text{dist}_{\Delta_q - C}(y, \tau) = q$. \square

Remark 1. *By an inductive proof like the one of Lemma 3, one can easily show that the maximum degree Δ of Δ_q is exactly $2 \cdot q$ for $q > 0$. Moreover, due to Lemma 3, the diameter of Δ_q is bounded in $O(q)$.*

Another observation on Δ_q is that any deletion of d edges increases the length of any shortest σ - τ path to at most $d + 1$, unless the edge deletion forms a σ - τ edge cut.

Lemma 4. *Let $D \subseteq E(\Delta_q)$ be a subset of edges of Δ_q . If D is not a σ - τ edge cut, then there is a σ - τ path of length at most $|D| + 1$ in $\Delta_q - D$.*

Proof. We prove the statement of the lemma by induction on q . For the induction base with $q = 0$, observe that since D is not a σ - τ edge cut, it follows that $D = \emptyset$ and, hence, σ and τ have distance one.

For the induction step, assume that the statement of the lemma is true for Δ_{q-1} . Now, let $D \subseteq E(\Delta_q)$ be a subset of edges of Δ_q such that D is not a σ - τ edge cut. If $\{\sigma, \tau\} \notin D$, then there is a σ - τ path of length one and the statement of the lemma holds. Now consider the case $\{\sigma, \tau\} \in D$. Denote by u the (unique) vertex that is adjacent to the two special vertices σ and τ . If $\{\sigma, \tau\} \in D$, then every σ - τ path in $\Delta_q - D$ contains u and hence $\text{dist}_{\Delta_q - D}(\sigma, \tau) = \text{dist}_{\Delta_q - D}(\sigma, u) + \text{dist}_{\Delta_q - D}(u, \tau)$. (If there is no σ - u -path or no u - τ -path in $\Delta_q - D$, then D is a σ - τ edge cut; a contradiction to the assumption of the lemma.) Now let Δ'_{q-1} and Δ''_{q-1} be the two $(q-1)$ -T-subfractals of Δ_q , so that Δ'_{q-1} (Δ''_{q-1}) has the special vertices σ and u (u and τ). It follows that D can be partitioned into $D = D' \cup D'' \cup \{\sigma, \tau\}$ with $D' \subseteq E(\Delta'_{q-1})$ and $D'' \subseteq E(\Delta''_{q-1})$. By induction hypothesis, it follows that there is a σ - u path of length at most $|D'| + 1$ in $\Delta'_{q-1} - D'$ and a u - τ path of length at most $|D''| + 1$ in $\Delta''_{q-1} - D''$. Hence, there is a σ - τ path of length at most $|D'| + |D''| + 2 = |D| + 1$ in $\Delta_q - D$. \square

By Lemma 4, the distance of the two special vertices σ and τ is upper-bounded by the number of edge deletions, where the deleted edges do not form a σ - τ edge cut. Hence, if only few edges are deleted in Δ_q , then σ and τ are not far away from each other. The next lemma generalizes this by stating that the distance of *any* vertex in Δ_q to σ or to τ is quite small, even if a few edges are deleted. Here “quite small” means that if $O(q)$ edges are deleted, then the distance is still $O(q)$ which is logarithmic in the size of Δ_q .

Lemma 5. *Let $D \subseteq E(\Delta_q)$ be a subset of edges of Δ_q and let x be an arbitrary vertex in $V(\Delta)$.*

- (A) *If $\Delta_q - D$ is connected, then $\text{dist}_{\Delta_q - D}(\sigma, x) \leq q + |D| + 1$ for all $x \in V(\Delta_q)$.*
- (B) *If $\Delta_q - D$ has exactly two connected components, with σ and τ being in different components, then $\min_{z \in \{\sigma, \tau\}} \{\text{dist}_{\Delta_q - D}(z, x)\} \leq q + |D| - 1$ for all $x \in V(\Delta_q)$.*

Proof. We prove the two statements (A) and (B) simultaneously with an induction on depth q of the T-fractal.

The base case is $q = 0$. For statement (A), observe that $D = \emptyset$. Thus, since τ has distance one to σ , statement (A) follows. For statement (B), observe that $D = \{\{\sigma, \tau\}\}$. Thus, statement (B) holds.

As our induction hypothesis, we assume that (A) and (B) hold for $1, \dots, q-1$. We write IH.(A) and IH.(B) for the induction hypothesis of (A) and (B), respectively. We introduce some notation used for the induction step for both statements. Let Δ_q , $q > 0$, the T-fractal with special vertices σ and τ and let u be the (unique) vertex in Δ_q that is adjacent to σ and τ , that is, u is on the boundary B_1 of Δ_q . Denote with Δ'_{q-1} (Δ''_{q-1}) the left (right) subfractal of Δ_q with special vertices σ and u (u and τ). Furthermore, let D' (D'') be the subset of edges of D deleted in Δ'_{q-1} (Δ''_{q-1}).

For the inductive step, we consider the two cases $\{\sigma, \tau\} \notin D$ and $\{\sigma, \tau\} \in D$.

Case 1: $\{\sigma, \tau\} \notin D$. Obviously, this case excludes (B), since σ and τ are in the same connected component. Thus, we consider the induction step for (A). Let x be in the left subfractal Δ'_{q-1} . If D' does not form an edge cut in Δ'_{q-1} , then by IH.(A) it follows that $\text{dist}_{\Delta'_{q-1} - D'}(\sigma, x) \leq q - 1 + |D'| + 1 \leq q + |D|$. Thus, we consider the case where D' forms an edge cut in Δ'_{q-1} . Observe that such an edge cut fulfills the requirements of statement (B) for Δ'_{q-1} . By IH.(B), it follows that $\min_{z \in \{\sigma, u\}} \{\text{dist}_{\Delta'_{q-1} - D'}(z, x)\} \leq q - 1 + |D'| - 1 < q + |D|$. If $z = \sigma$, then we are done. Thus let $z = u$, where u is the vertex incident to both σ and τ in Δ_q . We know that $\Delta_q - D$ is connected, and thus there exists an u - τ path in the right subfractal $\Delta''_{q-1} - D''$. By Lemma 4, it follows that $\text{dist}_{\Delta''_{q-1} - D''}(u, \tau) \leq |D''| + 1$. Recall that $\{\sigma, \tau\} \notin D$. In total, we get

$$\begin{aligned} \text{dist}_{\Delta_q - D}(x, \sigma) &\leq \text{dist}_{\Delta'_{q-1} - D'}(u, x) + \text{dist}_{\Delta''_{q-1} - D''}(u, \tau) + 1 \\ &\leq q - 1 + |D'| - 1 + |D''| + 1 + 1 = q + |D|. \end{aligned}$$

In the cases, we obtain that $\text{dist}_{\Delta_q - D}(x, \sigma) \leq q + |D|$ and hence, $\text{dist}_{\Delta_q - D}(x, \tau) \leq q + |D| + 1$. In case that x is in the right subfractal Δ''_{q-1} , it follows by symmetry that $\text{dist}_{\Delta_q - D}(x, \tau) \leq q + |D|$ and hence, $\text{dist}_{\Delta_q - D}(x, \sigma) \leq q + |D| + 1$.

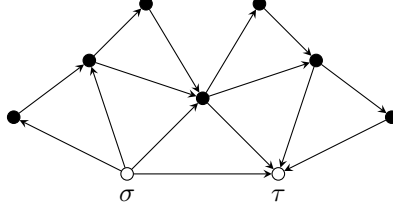


Figure 4: The directed T-fractal $\vec{\Delta}_3$.

Case 2: $\{\sigma, \tau\} \in D$. First, we consider the step for statement (A). Let x be in the left subfractal Δ'_{q-1} . Observe that D' does not form an edge cut in Δ'_{q-1} , since otherwise the graph is not connected. Thus, $\Delta'_{q-1} - D'$ is connected, and by IH.(A) it follows that $\text{dist}_{\Delta'_{q-1}-D'}(\sigma, x) \leq q - 1 + |D'| + 1 < q + |D|$.

Now, let x be in the right subfractal Δ''_{q-1} . Again, D'' does not form an edge cut in Δ''_{q-1} . By IH.(A), $\text{dist}_{\Delta''_{q-1}-D''}(u, x) \leq q - 1 + |D''| + 1$. Since u and σ are connected in $\Delta'_{q-1} - D'$, we can apply Lemma 4 on u and σ . In total, with $D = D' \cup D'' \cup \{\sigma, \tau\}$ we get:

$$\begin{aligned} \text{dist}_{\Delta_q-D}(x, \sigma) &\leq \text{dist}_{\Delta'_{q-1}-D'}(u, \sigma) + \text{dist}_{\Delta''_{q-1}-D''}(u, x) \\ &\leq q - 1 + |D'| + 1 + |D''| + 1 \leq q + |D|. \end{aligned}$$

Next, we consider the step for statement (B). Observe that the edge cut formed by edges in D cannot form edge cuts in Δ'_{q-1} and in Δ''_{q-1} at the same time since otherwise there are more than two connected components. Let x be in the left subfractal and let edges in D' do not form an edge cut in $\Delta'_{q-1} - D'$. Then, by IH.(A), it follows that $\text{dist}_{\Delta'_{q-1}-D'}(\sigma, x) \leq q - 1 + |D'| + 1 \leq q + |D| - 1$. Thus, let the edges in D' form an edge cut in $\Delta'_{q-1} - D'$. By IH.(B), either $\text{dist}_{\Delta'_{q-1}-D'}(\sigma, x) \leq q - 1 + |D'| - 1 < q + |D| - 1$, or $\text{dist}_{\Delta'_{q-1}-D'}(u, x) \leq q - 1 + |D'| - 1$. For the latter case, recall that the edges in D'' do not form a cut in Δ''_{q-1} , that is, $\Delta''_{q-1} - D''$ is connected. By Lemma 4, it follows that $\text{dist}_{\Delta''_{q-1}-D''}(u, \tau) \leq |D''| + 1$. In total, we get:

$$\begin{aligned} \text{dist}_{\Delta_q-D}(x, \tau) &\leq \text{dist}_{\Delta'_{q-1}-D'}(x, u) + \text{dist}_{\Delta''_{q-1}-D''}(u, \tau) \\ &\leq q - 1 + |D'| - 1 + |D''| + 1 < q + |D| - 1. \end{aligned}$$

The case where x is in the right subfractal follows by symmetry. \square

3.2 Directed Variants of T-Fractals

By definition, a T-fractal is an undirected graph. We now discuss how to turn it into a directed graph, more precisely, into a directed acyclic graph. We denote the directed variant of Δ_q by $\vec{\Delta}_q$. We obtain $\vec{\Delta}_q$ from Δ_q as follows: Recall that each boundary forms a σ - τ path. For each boundary, we direct the edges in the boundary from σ to τ . By this, the obtained boundary forms a directed σ - τ path. Observe that σ has no incoming arcs, and the out-degree of σ equals $q + 1$. Further observe that τ has no outgoing arcs, and the in-degree of τ equals $q + 1$. Moreover, $\vec{\Delta}_q$ is acyclic, see Fig. 4 for an illustration.

Except for Lemma 5, all results from Section 3.1 can be transferred to $\vec{\Delta}_q$. Lemma 1 and Corollary 1 hold since we still have the same degree on σ and τ and the boundaries still form disjoint (directed) σ - τ paths. Furthermore, we still have the equivalent recursive definition with the adjustment that the edge between σ and τ becomes an arc from σ to τ . We define the dual structure of $\vec{\Delta}_q$ as the dual structure of the underlying undirected variant Δ_q . By this, it is not hard to adapt Lemmas 2 and 3. For the latter result, and additionally for Lemma 4, we make use of the fact that in the undirected case, we traverse the edges of the undirected Δ_q in the same direction as they are directed in $\vec{\Delta}_q$.

Regarding an equivalent of Lemma 5 for the directed variant, with small effort one can modify the proof of Lemma 5 to show the following.

Lemma 6. *Let $D \subseteq E(\vec{\Delta}_q)$ be a subset of arcs of $\vec{\Delta}_q$ and let x be an arbitrary vertex in $V(\vec{\Delta}_q)$. If $x \in V(\vec{\Delta}_q)$ is reachable from σ in $\vec{\Delta}_q - D$, then $\text{dist}_{\vec{\Delta}_q - D}(\sigma, x) \leq q + |D| + 1$.*

Proof. We prove the statement with an induction on depth q of the T-fractal.

The base case is $q = 0$. If $x = \sigma$, the statement immediately holds. If $x = \tau$, observe that $D = \emptyset$, since x is reachable from σ . Thus, since τ has distance one to σ , the statement follows.

As our induction hypothesis, we assume that the statement holds for $1, \dots, q-1$. We introduce some notation used for the induction step. Let $\vec{\Delta}_q$, $q > 0$, be the directed T-fractal with special vertices σ and τ and let u be the (unique) vertex in $\vec{\Delta}_q$ that is adjacent to σ and τ , that is, u is on the boundary B_1 of $\vec{\Delta}_q$. Denote with $\vec{\Delta}'_{q-1}$ ($\vec{\Delta}''_{q-1}$) the left (right) subfractal of $\vec{\Delta}_q$ with special vertices σ and u (u and τ). Furthermore, let D' (D'') be the subset of arcs of D deleted in $\vec{\Delta}'_{q-1}$ ($\vec{\Delta}''_{q-1}$).

Let $x \in V(\vec{\Delta})$ be an arbitrary vertex reachable from σ in $\vec{\Delta}_q - D$. For the inductive step, we consider the two cases of the position of x in $\vec{\Delta}_q$.

Case 1: x appears in $\vec{\Delta}'_{q-1}$. By induction hypothesis, it follows that

$$\text{dist}_{\vec{\Delta}_q - D}(\sigma, x) = \text{dist}_{\vec{\Delta}'_{q-1} - D'}(\sigma, x) \leq q - 1 + |D'| + 1 \leq q + |D|.$$

Case 2: x appears in $\vec{\Delta}''_{q-1}$. Since x is reachable from σ and τ has no outgoing arcs, it follows that u is reachable from σ as well. By the version of Lemma 4 for directed T-fractals, it follows that $\text{dist}_{\vec{\Delta}'_{q-1} - D'}(\sigma, u) \leq |D'| + 1$. Together with the induction hypothesis, it follows that

$$\begin{aligned} \text{dist}_{\vec{\Delta}_q - D}(\sigma, x) &\leq \text{dist}_{\vec{\Delta}'_{q-1} - D'}(\sigma, u) + \text{dist}_{\vec{\Delta}''_{q-1} - D''}(u, \tau) \\ &\leq |D'| + 1 + q - 1 + |D''| + 1 \leq q + |D| + 1. \end{aligned}$$

□

Observe that the case that x reaches τ is symmetric.

3.3 Application Manual for T-Fractals

The aim of this subsection is to provide two general guidelines on how to use T-fractals in cross-compositions. To this end, we introduce two general constructions—one for undirected graphs and one for directed graphs. We start with the undirected case.

Construction 1. *Given $p = 2^q$ instances $\mathcal{I}_1, \dots, \mathcal{I}_p$ of an NP-hard graph problem, where each instance \mathcal{I}_i has a unique source vertex s_i and a unique sink vertex t_i .*

- (i) Equip Δ_q^c with some “appropriate” edge cost $c \in \mathbb{N}$.
- (ii) Let v_0, \dots, v_p be the vertices of the boundary B_q , labeled by their distances to σ in the σ - τ path corresponding to B_q (observe that $v_0 = \sigma$ and $v_p = \tau$).
- (iii) Incorporate each of the p graphs of the input instances into Δ_q^c as follows: for each $i \in [p]$, merge s_i with vertex v_{i-1} in Δ_q^c and merge t_i with v_i in Δ_q^c .

Refer to Fig. 5 for an illustrative example of Construction 1. In Construction 1, the T-fractal works as an instance selector by deleting edges corresponding to a minimum edge cut, which, by Lemma 1, is of size $q+1$. Hence, each minimum edge cut costs $c \cdot (q+1)$. The idea is that if we choose an appropriate value for c (larger than the budget in the instances $\mathcal{I}_1, \dots, \mathcal{I}_p$) and an appropriate budget in the composed instance (e.g. $c \cdot (q+1)$ plus the budget in the instances $\mathcal{I}_1, \dots, \mathcal{I}_p$), then we can only afford to delete at most $q+1$ edges in Δ_q^c . Furthermore, if the at most $q+1$ edges chosen to be deleted do not form a minimum σ - τ edge cut in Δ_q^c , then, by Lemma 4, the shortest σ - τ path has length at most $q+2$. Thus, by requiring in the composed instance that σ and τ have distance more than $q+2$, we enforce that any solution for the composed instance contains a minimum σ - τ edge cut in Δ_q^c . By Lemma 2, each such minimum edge cut corresponds to one root-leaf path in the dual structure T_q of Δ_q^c . Observe that each leaf in the dual structure of Δ_q^c one-to-one corresponds to an attached source instance. Hence, with an appropriate choice of c , the budget in the composed instance, and the required distance between σ and τ , the T-fractal ensures

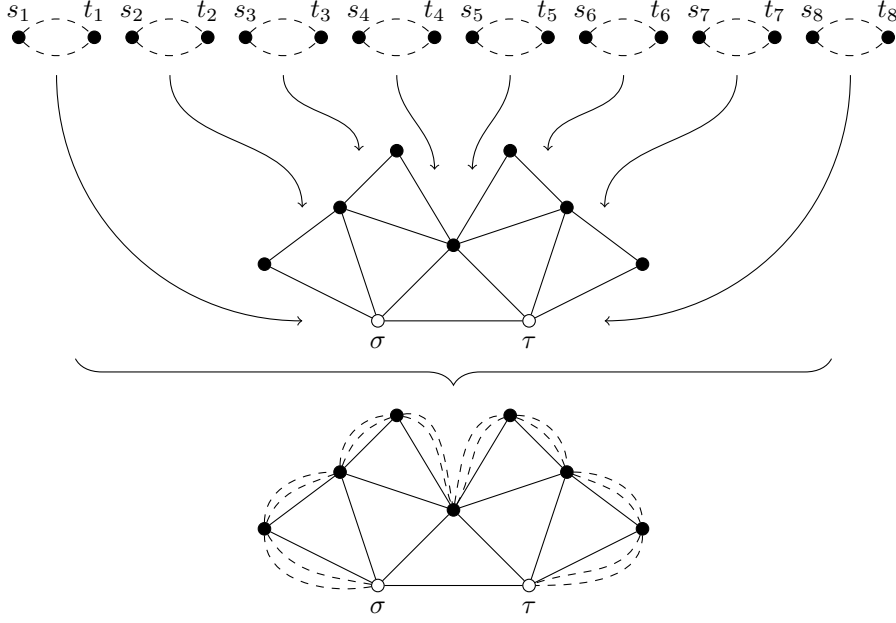


Figure 5: Illustration of Construction 1 with $p = 2^3 = 8$. The vertices s_1, \dots, s_8 indicate the source vertices in the eight input instances, and t_1, \dots, t_8 indicate the sink vertices in the eight input instances. We use dashed lines to sketch the input graphs. Below the curved brace, the resulting graph of the target instance is sketched.

that one instance is “selected”. We say that a minimum σ - τ edge cut in Δ_q^c *selects* an instance \mathcal{I} if the edge cut corresponds to the root-leaf path with the leaf corresponding to instance \mathcal{I} .

Observation 2. *Every minimum edge cut C in Δ_q^c selects exactly one instance \mathcal{I} . Conversely, every instance \mathcal{I} can be selected by exactly one minimum edge cut.*

Using the same ideas as above and transferring them to the directed case yields the following construction with analogous properties.

Construction 2. *Given $p = 2^q$ instances $\mathcal{I}_1, \dots, \mathcal{I}_p$ of an NP-hard problem on directed acyclic graphs, where each instance \mathcal{I}_i has a unique source vertex s_i and a unique sink vertex t_i .*

- (i) *Equip $\vec{\Delta}_q^c$ with some “appropriate” edge cost $c \in \mathbb{N}$, where σ is the vertex with no incoming arc.*
- (ii) *Let v_0, \dots, v_p be the vertices of the boundary B_q , labeled by their distances to σ in the σ - τ path corresponding to B_q (observe that $v_0 = \sigma$ and $v_p = \tau$).*
- (iii) *Incorporate each of the p directed acyclic graphs of the input instances into $\vec{\Delta}_q^c$ as follows: for each $i \in [p]$, merge s_i with vertex v_{i-1} in $\vec{\Delta}_q^c$ and merge t_i with vertex v_i in $\vec{\Delta}_q^c$.*

In the rest of the paper, we use Constructions 1 and 2 in OR-cross-compositions to rule out the existence of polynomial kernels. We baptize this approach *fractalism*. In particular, we provide the source and the target problem, appropriate values for the edge cost c and the budget in the composed instance, and the required distance between the special vertices σ and τ . Observe that the directed graph obtained from Construction 2 is acyclic. Hence, by Construction 2 we can apply OR-cross-compositions for problems on directed acyclic graphs. We remark that there is a third construction where we drop the “acyclicity” requirement in Construction 2. This yields a construction of a directed, possibly cyclic graph. In this sense, Construction 2 is a special case of the third construction.

4 Applications to Length-Bounded Cut Problems

In this section, we rule out the existence of polynomial kernels for several problems (and their variants) under the assumption that $\text{NP} \not\subseteq \text{coNP}/\text{poly}$. To this end, we combine the framework of OR-cross-compositions with our fractalism technique as described in Section 3.3.

4.1 Length-Bounded Edge-Cut

Our first fractalism application is the LENGTH-BOUNDED EDGE-CUT problem [2], also known as the problem of finding bounded edge undirected cuts [21], or the SHORTEST PATH MOST VITAL EDGES problem [4, 28].

LENGTH-BOUNDED EDGE-CUT (LBEC)

Input: An undirected graph $G = (V, E)$, with $s, t \in V$, and two integers k, ℓ .

Question: Is there a subset $F \subseteq E$ of cardinality at most k such that $\text{dist}_{G-F}(s, t) \geq \ell$?

The problem is NP-complete [25] and fixed-parameter tractable with respect to (k, ℓ) [21]. If k is at least the size of any s - t edge cut, then the problem becomes polynomial-time solvable by simply computing a minimum s - t edge cut. Thus, throughout this section, we assume that k is smaller than the size of any minimum s - t edge cut. The generalized problem where each edge is equipped with positive length remains NP-hard even on series-parallel and outerplanar graphs [2]. The directed variant with positive edge lengths remains NP-hard on planar graphs where the source and the sink vertex are incident to the same face [33]. Recently, Dvořák and Knop [17] showed that the problem can be solved in polynomial time on graphs of bounded treewidth. Here, we answer an open question [21] concerning the existence of a polynomial kernel with respect to the combined parameter (k, ℓ) .²

Theorem 1. *Unless $\text{NP} \subseteq \text{coNP}/\text{poly}$, LENGTH-BOUNDED EDGE-CUT parameterized by (k, ℓ) does not admit a polynomial kernel.*

Proof. We OR-cross-compose $p = 2^q$ instances of LBEC into one instance of LBEC(k', ℓ'). An instance $(G_i, s_i, t_i, k_i, \ell_i)$ of LBEC is called *bad* if $\max\{k_i, \ell_i\} > |E(G_i)|$ or $\min\{k_i, \ell_i\} < 0$. We define the polynomial equivalence relation \mathcal{R} on the instances of LBEC as follows: two instances $(G_i, s_i, t_i, k_i, \ell_i)$ and $(G_j, s_j, t_j, k_j, \ell_j)$ of LBEC are \mathcal{R} -equivalent if and only if $k_j = k_i$ and $\ell_j = \ell_i$, or both are bad instances. Clearly, the relation \mathcal{R} fulfills condition (i) of an equivalence relation (see Section 2). Observe that the number of equivalence classes of a finite set of instances is upper-bounded by the maximal size of a graph over the instances, hence condition (ii) holds. Thus, we consider p \mathcal{R} -equivalent instances $\mathcal{I}_i := (G_i, s_i, t_i, k, \ell)$, $i = 1, \dots, p$. We remark that we can assume that $\ell \geq 3$, since otherwise LBEC is solvable in polynomial time by counting all edges connecting the source with the sink vertex. We OR-cross-compose into one instance $\mathcal{I} := (G, s, t, k', \ell')$ of LBEC(k', ℓ') with $k' = k^2 \cdot (\log(p) + 1) + k$ and $\ell' = \ell + \log(p)$ as follows.

Construction: Apply Construction 1 with edge cost $c = k^2$. In addition, set $s := \sigma$ and $t := \tau$. Let G denote the obtained graph.

Correctness: We show that \mathcal{I} is a YES-instance if and only if there exists an $i \in [p]$ such that \mathcal{I}_i is a YES-instance.

“ \Leftarrow ”: Let $i \in [p]$ be such that \mathcal{I}_i is YES. Following Observation 2 in Section 3.3, let C be the minimum s - t cut in Δ_q^c that selects instance \mathcal{I}_i . Recall that C is of size $q + 1$ and that the edge cost equals k^2 . Thus, the minimum s - t cut C has cost $(q + 1) \cdot k^2 = (\log(p) + 1) \cdot k^2$.

Note that after deleting the edges in C , the vertices s and t are only connected via paths through the incorporated graph G_i . Since \mathcal{I}_i is YES, we can delete k edges (equal to the remaining budget) such that the distance of s_i and t_i in G_i is at least ℓ . Together with Lemma 3 in Section 3.1, such an additional edge deletion increases the length of any shortest s - t path in G to at least $\ell + \log(p) = \ell'$. Hence, \mathcal{I} is a YES-instance.

“ \Rightarrow ”: Suppose that one can delete at most k' edges in G such that each s - t path is of length at least ℓ' . Since the budget allows $\log(p) + 1$ edge-deletions in Δ_q^c , by Lemma 4 in Section 3.1, if

²The question also appeared in the list of open problems of the FPT School 2014, 17-22 August 2014, Będlewo, Poland, <http://fptschool.mimuw.edu.pl/opl.pdf>.

we do not cut s and t in Δ_q^c , then there is an s - t path of length $\log(p) + 2$. Since $\ell \geq 3$, such an edge deletion does not yield a solution. Thus, in every solution of \mathcal{I} , a subset of the deleted edges forms a minimum s - t edge cut in Δ_q^c and thus, by Observation 2, selects an input instance.

Consider an arbitrary solution to \mathcal{I} , that is, an edge subset of $E(G)$ of cardinality at most k' whose deletion increases the shortest s - t path to at least ℓ' . Let \mathcal{I}_i , $i \in [p]$, be the selected instance. Note that any shortest s - t path contains edges in the selected instance \mathcal{I}_i . By Lemma 3, we know that the length of the shortest s - s_i path and the length of the shortest t_i - t path sum up to exactly $\log(p)$. It follows that the remaining budget of k edge deletions is spent in G_i in such a way that there is no path from s_i to t_i of length smaller than ℓ in G_i . Hence, \mathcal{I}_i is a YES-instance. \square

Golovach and Thilikos [21] showed that LBEC on directed acyclic graphs is NP-complete. Using Construction 2 instead of Construction 1 with LBEC on directed acyclic graphs as source and target problem, the same argument as in the proof of Theorem 1 yields the following.

Theorem 2. *Unless $\text{NP} \subseteq \text{coNP} / \text{poly}$, LENGTH-BOUNDED EDGE-CUT on directed acyclic graphs parameterized by (k, ℓ) does not admit a polynomial kernel.*

In the following, we consider LBEC on planar graphs. To the best of our knowledge, it was not shown before whether LBEC remains NP-hard on planar graphs. This is what we state next.

Theorem 3. *LENGTH-BOUNDED EDGE-CUT is NP-hard even on planar undirected graphs as well as on planar directed acyclic graphs, where for both problems s and t are incident to the outer face.*

We defer the proof of Theorem 3 to the appendix (see Proof 1).

Due to Theorem 3, we can use LBEC on planar undirected graphs as well as on planar directed acyclic graphs, where in both cases the source and sink vertices are incident to the outer face, as source problem for OR-cross-compositions. The property that the source and the sink vertices are allowed to be incident with the same face in the input graph allows us to use Constructions 1 and 2 with a target problem on planar graphs. Hence, together with the same argumentation as in the proof of Theorem 1, we obtain the following.

Theorem 4. *Unless $\text{NP} \subseteq \text{coNP} / \text{poly}$, LENGTH-BOUNDED EDGE-CUT on planar undirected graphs as well as on planar directed acyclic graphs parameterized by (k, ℓ) do not admit a polynomial kernel.*

4.2 Minimum Diameter Edge Deletion

Our second fractalism application concerns a problem introduced by Schoone et al. [35].

MINIMUM DIAMETER EDGE DELETION (MDED)

Input: A connected, undirected graph $G = (V, E)$, two integers k, ℓ .

Question: Is there a subset $F \subseteq E$ of cardinality at most k such that $G - F$ is connected and $\text{diam}(G - F) \geq \ell$?

The problem was shown to be NP-complete, also on directed graphs [35]. A simple search tree algorithm yields fixed-parameter tractability with respect to (k, ℓ) :

Theorem 5. *MINIMUM DIAMETER EDGE DELETION can be solved in $O((\ell - 1)^k n^2(n + m))$ time.*

We defer the proof of Theorem 5 to the appendix (see Proof 2).

Complementing the fixed-parameter tractability of MDED(k, ℓ), we show the following.

Theorem 6. *Unless $\text{NP} \subseteq \text{coNP} / \text{poly}$, MINIMUM DIAMETER EDGE DELETION parameterized by (k, ℓ) does not admit a polynomial kernel.*

Proof. We OR-cross-compose $p = 2^q$ instances of LENGTH-BOUNDED EDGE-CUT (LBEC) on connected graphs into one instance of MDED(k, ℓ) as follows. Apply Construction 1 with $p = 2^q$ instances $\mathcal{I}_i := (G_i, s_i, t_i, k, \ell)$, $i = 1, \dots, p$, of the input problem LBEC on connected graphs, target problem MDED(k, ℓ), and edge cost $c = k^2$. The equivalence relation on the input instances is defined as in the proof of Theorem 1. Let $n_{\max} := \max_{i \in [p]} |V(G_i)|$. In addition, attach to σ as well as on τ a path of length $L := n_{\max} \cdot (2 \log(p) + 3) + 1$ each. Denote the endpoint of the

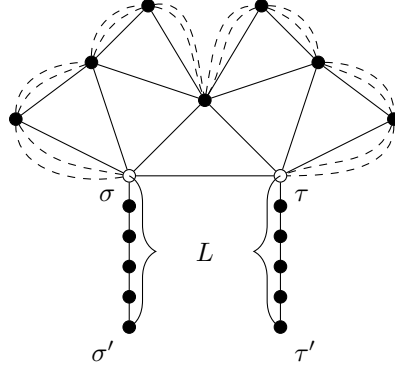


Figure 6: Cross-composition for MINIMUM DIAMETER EDGE DELETION(k, ℓ) with $p = 8 = 2^3$, and $L = 9 \cdot n_{\max} + 1$. Dashed lines sketch the boundaries of the graphs in the p input instances.

path attached to σ by σ' (where $\sigma' \neq \sigma$), and let τ' be defined analogously. Let G denote the obtained graph. Refer to Figure 6 for an exemplified illustration of the described construction. Let $\mathcal{I} := (G, k', \ell')$ be the target instance of MDDED(k', ℓ') with $k' = k^2 \cdot (\log(p) + 1) + k$ and $\ell' = 2 \cdot L + \log(p) + \ell$.

Correctness: We show that \mathcal{I} is a YES-instance for MDDED(k', ℓ') if and only if there exists an $i \in [p]$ such that \mathcal{I}_i is a YES-instance for LBEC on connected graphs.

“ \Leftarrow ”: Let $\mathcal{I}_i, i \in [p]$, be a YES-instance for LBEC on connected graphs. Following Observation 2, we delete all edges in the minimum cut in Δ_q^c that selects instance \mathcal{I}_i . Then, we delete edges corresponding to a solution for \mathcal{I}_i without disconnecting the graph G (observe that we can always find such a solution). Let $X \subseteq E(G)$ be the set of deleted edges. The distance of σ and τ in $G - X$ is at least $\log(p) + \ell$, and thus, the distance of σ' and τ' is at least $2 \cdot L + \log(p) + \ell = \ell'$. Hence, the diameter is at least ℓ' after k' edge deletions that leave the graph connected. It follows that \mathcal{I} is a YES-instance.

“ \Rightarrow ”: Conversely, suppose that \mathcal{I} allows k' edge deletions such that the remaining graph is connected and has diameter at least ℓ' . Let $X \subseteq E(G)$ be a solution. First observe that $G - X$ is connected. Consider the instances appended to the T-fractal as the artificial $q + 1$ st boundary of a $(q + 1)$ -T-fractal, where an edge in this boundary has length n_{\max} . Thus, we can apply Lemma 5(A) to this artificial $(q + 1)$ -T-fractal. Recall that our budget only allows $\log(p) + 1$ edge deletions (of cost k^2) in Δ_q^c . Hence we get that the distance to σ (and by symmetry to τ) of every vertex contained either in Δ_q^c or in any appended instance is at most $n_{\max} \cdot (\log(p) + \log(p) + 3) = L - 1$. It follows that $\text{dist}_{G-X}(x, \sigma) \leq \text{dist}_{G-X}(\sigma, \sigma')$ and $\text{dist}_{G-X}(x, \tau) \leq \text{dist}_{G-X}(\tau, \tau')$ for all $x \in V(G)$. Moreover, for all $x, y \in V(G)$ we have:

$$\begin{aligned} \text{dist}_{G-X}(x, y) &\leq \text{dist}_{G-X}(x, \sigma) + \text{dist}_{G-X}(\sigma, \tau) + \text{dist}_{G-X}(\tau, y) \\ &\leq \text{dist}_{G-X}(\sigma', \sigma) + \text{dist}_{G-X}(\sigma, \tau) + \text{dist}_{G-X}(\tau, \tau') \\ &= \text{dist}_{G-X}(\sigma', \tau'). \end{aligned}$$

Hence, σ', τ' is the pair of vertices with the largest distance in $G - X$ and, thus, $\text{dist}_{G-X}(\sigma', \tau') \geq \ell'$. Observe that $\text{dist}_{G-X}(\sigma', \tau') \geq \ell'$ if and only if $\text{dist}_{G-X}(\sigma, \tau) \geq \log(p) + \ell$ since every shortest σ' - τ' path contains both σ and τ . Following the argumentation in the correctness proof of Theorem 1, it follows that there is an instance $\mathcal{I}_i, i \in [p]$, that is a YES-instance for LBEC on connected graphs. \square

In their NP-hardness-proof for MDDED, Schoone et al. [35] reduce from HAMILTONIAN PATH (HP) to MDDED. The reduction does not modify the graph, that is, the input graph for HP remains the same for the MDDED instance. Since HP remains NP-hard on planar graphs [20], the reduction of Schoone et al. implies that MDDED is NP-hard even on planar graphs. Due to Theorem 3, using Construction 1 with LBEC on planar graphs as source problem, and MDDED(k, ℓ) on planar graphs as target problem, we obtain the following.

Theorem 7. *Unless $\text{NP} \subseteq \text{coNP} / \text{poly}$, MINIMUM DIAMETER EDGE DELETION on planar graphs parameterized by (k, ℓ) does not admit a polynomial kernel.*

The diameter of a directed graph is defined as the maximum length of a shortest directed path over any two vertices in any order. The diameter of a directed graph that is not strongly-connected equals infinity. Thus, MINIMUM DIAMETER EDGE DELETION on directed graphs is defined as follows: given a strongly-connected directed graph $G = (V, E)$, and two integers k and ℓ , the question is whether there is a subset $F \subseteq E$ of cardinality at most k such that $G - F$ is strongly-connected and $\text{diam}(G - F) \geq \ell$? Observe that MINIMUM DIAMETER EDGE DELETION on directed planar graphs parameterized by (k, ℓ) is FPT, as a consequence of the proof of Theorem 5.

Theorem 8. *Unless $\text{NP} \subseteq \text{coNP} / \text{poly}$, MINIMUM DIAMETER EDGE DELETION on directed planar graphs parameterized by (k, ℓ) does not admit a polynomial kernel.*

Proof (Sketch). The following proof adapts the ideas of the proof of Theorem 6. Thus, we highlight the differences to the proof instead of providing a full proof here.

We OR-cross-compose $p = 2^q$ instances of LENGTH-BOUNDED EDGE-CUT (LBEC) on planar, directed acyclic graphs into one instance of MDDED(k, ℓ) on directed planar graphs. We assume without loss of generality that in each graph of the input instances, the source reaches every vertex, and every vertex reaches the sink.

We apply Construction 2 with the following additions. Let n_{\max} and k' be defined as in the proof of Theorem 6, that is, $n_{\max} := \max_{i \in [p]} |V(G_i)|$ and $k' := k^2 \cdot (\log(p) + 1) + k$. Let $L := \ell \cdot n_{\max} \cdot (2 \log(p) + 3) + 1$ and $\ell' = 2 \cdot L + \log(p) + \ell$. Attach to σ as well as to τ a path of length L each. Denote the endpoint of the path attached to σ by σ' (where $\sigma' \neq \sigma$), and let τ' be defined analogously. Direct all edges in the paths towards from σ' to σ and from τ to τ' respectively. Moreover, add to the graph the arc (τ', σ') , and the arc (τ, σ) , the latter with cost $k' + 1$.

Next, we adjust the instances we compose in order to ensure that we can delete all the arcs we want without destroying the property that the source reaches every vertex and every vertex reaches the sink. Let G_i be the graph in instance \mathcal{I}_i for each $i \in [p]$. For each arc $(v, w) \in E(G_i)$, connect v and w by an additional path of length ℓ directed towards w . Apply this for each G_i , $i \in [p]$, and let G'_i the graph obtained from graph G_i . Note that the directed graph G'_i remains planar and acyclic. Observe that none of the introduced arcs will be in a minimal solution for the LBEC instance since they only occur in paths of length ℓ . Hence, \mathcal{I}_i is a YES-instance of LBEC on planar, directed acyclic graphs if and only if $(G'_i, s_i, t_i, k, \ell)$ is a YES-instance of LBEC on planar, directed acyclic graphs. Furthermore, in the composed MINIMUM DIAMETER EDGE DELETION-instance, none of the introduced arcs will be deleted as this would introduce a vertex without in-going or without out-going arcs and this is not allowed in the problem setting.

Let G denote the obtained graph. Observe that G is planar, directed and strongly-connected.

Suppose (G, k', ℓ') is a YES-instance of MINIMUM DIAMETER EDGE DELETION. Consider a solution $X \subseteq E(G)$ for the instance (G, k', ℓ') of MINIMUM DIAMETER EDGE DELETION on directed planar graphs. The crucial observation is that for any two vertices x, y not contained in the attached paths with endpoints σ' on the one, and τ' on the other hand, the following holds: $\max\{\text{dist}_{G-X}(x, y), \text{dist}_{G-X}(y, x)\} \leq \text{dist}_{G-X}(\sigma', \tau')$. To see this, note that the arc (τ, σ) has cost $k' + 1$ and thus $(\tau, \sigma) \notin X$. Since G is strongly-connected, both x and y are reachable and reach σ and τ . Moreover, σ is reachable from τ via the arc (τ, σ) . Without loss of generality, let $\text{dist}_{G-X}(x, y) = \max\{\text{dist}_{G-X}(x, y), \text{dist}_{G-X}(y, x)\}$. It holds that

$$\begin{aligned} \text{dist}_{G-X}(x, y) &\leq \text{dist}_{G-X}(x, \tau) + \text{dist}_{G-X}(\tau, \sigma) + \text{dist}_{G-X}(\sigma, y) \\ &\leq \ell \cdot n_{\max} \cdot (2 \log(p) + 2) + 1 + \ell \cdot n_{\max} \cdot (2 \log(p) + 2) \\ &= 2 \cdot \ell \cdot n_{\max} \cdot (2 \log(p) + 2) + 1 < \ell'. \end{aligned}$$

Herein, recall that we allow $\log(p)+1$ arc deletions in $\vec{\Delta}_q$. The second inequality follows from Lemma 6 and the fact that in each graph $G_i - X$ the vertex s_i has distance at most $\ell \cdot n_{\max}$ to t_i .

As a consequence, the vertices at distance ℓ' appear in the paths appended on σ and τ . Among them, note that $\text{dist}_{G-X}(\sigma', \tau')$ is maximal. Following the discussion in the proof of Theorem 6, the budget has to be spend in such a way that the arc-deletions form an σ - τ arc-cut in $\vec{\Delta}_q$, and

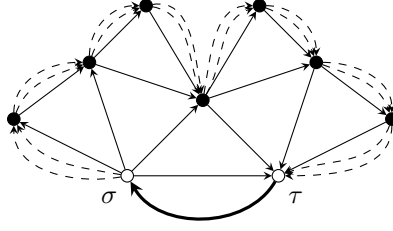


Figure 7: Cross-composition for $\text{DSCT}(k, \ell)$ with $p = 8 = 2^3$. Dashed lines sketch the boundaries of the graphs in the p input instances.

the remaining budget must be spend in such a way that the instance \mathcal{I}_i chosen by the cut allows no s_i - t_i path of length smaller than ℓ . Hence, the \mathcal{I}_i is a YES-instance.

Conversely, let \mathcal{I}_i be a YES-instance of LBEC on planar, directed acyclic graphs and let $X' \subseteq E(G_i)$ a minimum size solution. We added to each arc of G_i a directed path of length ℓ and, as discussed above, none of the arcs in these paths is in X' . Hence, in $G_i - X'$ every vertex is still reachable from s_i and reaches t_i . Deleting in G the arcs in X' and the arcs corresponding to the cut choosing \mathcal{I}_i preserves the strongly-connectivity of G . Let $X \subseteq E(G)$ be the set of deleted arcs. Following the discussion in the proof of Theorem 6, $\text{dist}_{G-X}(\sigma', \tau') \geq \ell'$. It follows that \mathcal{I} is a YES-instance of MINIMUM DIAMETER EDGE DELETION on directed planar graphs. \square

4.3 Directed Small Cycle Transversal

Our third fractalism application concerns the following problem.

DIRECTED SMALL CYCLE TRANSVERSAL (DSCT)

Input: A directed graph $G = (V, E)$, two integers k, ℓ .

Question: Is there a subset $F \subseteq E$ of cardinality at most k such that there is no induced directed cycle of length at most ℓ in $G - F$?

The problem is NP-hard [23], also on undirected graphs [41]. The NP-completeness of DSCT follows by a simple reduction from k -FEEDBACK ARC SET with an n -vertex graph, where we set $\ell = n$ and leave the graph unchanged in the reduction. We remark that the problem is also known as CYCLE TRANSVERSAL [8], or ℓ -(DIRECTED)-CYCLE TRANSVERSAL [23]. The undirected variant is also known as SMALL CYCLE TRANSVERSAL [39, 40].

As for the MINIMUM DIAMETER EDGE DELETION problem, there is a simple search tree algorithm showing fixed-parameter tractability with respect to (k, ℓ) .

Theorem 9. DIRECTED SMALL CYCLE TRANSVERSAL can be solved in $O(\ell^k \cdot (n + m))$ time.

We defer the proof of Theorem 9 to the appendix (see Proof 3).

Theorem 10. Unless $\text{NP} \subseteq \text{coNP} / \text{poly}$, DIRECTED SMALL CYCLE TRANSVERSAL parameterized by (k, ℓ) does not admit a polynomial kernel.

Proof. We OR-cross-compose $p = 2^q$ \mathcal{R} -equivalent instances of LBEC on directed acyclic graphs into one instance of $\text{DSCT}(k, \ell)$ as follows, where \mathcal{R} is defined as in the proof of Theorem 1. Recall that LENGTH-BOUNDED EDGE-CUT (LBEC) on directed acyclic graphs is NP-complete.

Construction: We apply Construction 2 with edge cost k^2 . In addition, we add the edge (τ, σ) with edge cost $k' + 1$, where $k' = k^2 \cdot (\log(p) + 1) + k$. We denote by G the obtained graph. We refer to Fig. 7 for an exemplified illustration of the construction. Observe that G is not acyclic, and the edge (τ, σ) participates in every cycle in G , that is, G without edge (τ, σ) is acyclic. Let (G, k', ℓ') be the target instance of $\text{DSCT}(k, \ell)$ with $\ell' = \ell + \log(p) + 1$.

Correctness: Note that every cycle in G uses the edge (τ, σ) . Since its edge cost equals $k' + 1$, the budget does not allow its deletion. Thus, the crucial observation is that the length of any shortest path from σ to τ must be increased to at least $\ell + \log(p) = \ell' - 1$. Hence, the correctness proof follows from the proof of Theorem 2. \square

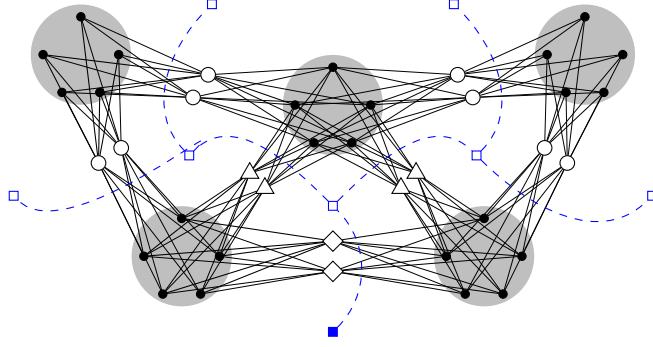


Figure 8: The vertex deletion variant $\Delta_2^{2;5}$ of T-fractals. Vertex types: empty diamonds belong to the boundary B_0 , empty triangles belong to the boundary B_1 , empty circles belong to the boundary B_2 . The squares and dashed lines indicate the dual structure, where the filled square corresponds to the root. We highlighted vertices in gray-filled circles that correspond to the vertices in the edge-deletion variant Δ_2 .

Due to Theorem 3, using LBEC on planar directed acyclic graphs as source problem in the proof of Theorem 10, we obtain the following.

Theorem 11. *Unless $\text{NP} \subseteq \text{coNP} / \text{poly}$, DIRECTED SMALL CYCLE TRANSVERSAL on planar directed graphs parameterized by (k, ℓ) does not admit a polynomial kernel.*

Remarkably, DSCT(k, ℓ) on planar undirected graphs admits a polynomial kernel [39].

5 Conclusion

We start with briefly sketching how our technique can be adapted such that it also applies to the vertex deletion (instead of edge deletion) versions of the considered problems. Afterwards, we discuss future challenges and open problems.

Extension to Vertex-Deletion Variants. Most of our results can be transferred to the vertex deletion variants of the considered edge deletion problems as follows. To this end, we modify the T-fractal as displayed in Fig. 8: First, subdivide each edge. Then, replace each vertex v in the original T-fractal by many pairwise non-adjacent vertices with the same neighborhood as v . The number of these introduced “false twins” is larger than the given budget such that the only way to disconnect vertices in the new fractal will be to delete vertices introduced from the subdivision of the edges. In this way, deleting a vertex in the new T-fractal corresponds to deleting an edge in the original fractal. This new fractal might not be planar anymore, but, as in the edge deletion variant, one can direct the edges in such a way that the obtained directed graph is acyclic.

We claim that the new T-fractal can be used in the same way as the original T-fractal in order to exclude polynomial kernels for vertex deletion variants of the problems discussed in this work—both in undirected and directed, but not for planar variants.

Outlook. We provided several case studies where our fractalism technique applies. It remains open to further explore the limitations and possibilities of our technique in more contexts. Table 1 in Section 1 presents an open question which should be clarified. Moreover, we could not settle the cases for vertex deletion problems when the underlying graphs are planar.

Acknowledgement. We thank Manuel Sorge (TU Berlin) for his help with the proof of Theorem 3.

References

- [1] Giorgio Ausiello, Pierluigi Crescenzi, Giorgio Gambosi, Viggo Kann, Alberto Marchetti-Spaccamela, and Marco Protasi. *Complexity and Approximation: Combinatorial Optimization Problems and Their Approximability Properties*. Springer, 1999. 2
- [2] Georg Baier, Thomas Erlebach, Alexander Hall, Ekkehard Köhler, Petr Kolman, Ondrej Pangrác, Heiko Schilling, and Martin Skutella. Length-bounded cuts and flows. *ACM Transactions on Algorithms*, 7(1):4, 2010. 12
- [3] Cristina Bazgan, Morgan Chopin, Marek Cygan, Michael R. Fellows, Fedor V. Fomin, and Erik Jan van Leeuwen. Parameterized complexity of firefighting. *Journal of Computer and System Sciences*, 80(7):1285–1297, 2014. 3
- [4] Cristina Bazgan, André Nichterlein, and Rolf Niedermeier. A refined complexity analysis of finding the most vital edges for undirected shortest paths. In *Proc. of the 9th International Conference on Algorithms and Complexity (CIAC 2015)*, volume 9079 of *LNCS*, pages 47–60. Springer, 2015. 2, 12
- [5] Michael A. Bekos, Martin Gronemann, and Chrysanthi N. Raftopoulou. Two-page book embeddings of 4-planar graphs. In *Proc. of the 31st International Symposium on Theoretical Aspects of Computer Science (STACS 2014)*, volume 25 of *LIPIcs*, pages 137–148. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2014. 20
- [6] René van Bevern, Robert Brederick, Morgan Chopin, Sepp Hartung, Falk Hüffner, André Nichterlein, and Ondřej Suchý. Parameterized complexity of dag partitioning. In *Proc. of the 8th International Conference on Algorithms and Complexity (CIAC '13)*, volume 7878 of *LNCS*, pages 49–60. Springer, 2013. 3
- [7] Hans L. Bodlaender, Rodney G. Downey, Michael R. Fellows, and Danny Hermelin. On problems without polynomial kernels. *Journal of Computer and System Sciences*, 75(8):423–434, 2009. 2
- [8] Hans L. Bodlaender, Fedor V. Fomin, Daniel Lokshtanov, Eelko Penninkx, Saket Saurabh, and Dimitrios M. Thilikos. (Meta) Kernelization. In *Proc. of the 50th Annual IEEE Symposium on Foundations of Computer Science (FOCS 2009)*, pages 629–638. IEEE Computer Society, 2009. 16
- [9] Hans L. Bodlaender, Stéphan Thomassé, and Anders Yeo. Kernel bounds for disjoint cycles and disjoint paths. *Theoretical Computer Science*, 412(35):4570–4578, 2011. 2
- [10] Hans L. Bodlaender, Bart M. P. Jansen, and Stefan Kratsch. Kernelization lower bounds by cross-composition. *SIAM Journal on Discrete Mathematics*, 28(1):277–305, 2014. 2, 4
- [11] Liming Cai, Jianer Chen, Rodney G. Downey, and Michael R. Fellows. Advice classes of parameterized tractability. *Annals of Pure and Applied Logic*, 84(1):119–138, 1997. 4
- [12] Marek Cygan, Fedor V. Fomin, Lukasz Kowalik, Daniel Lokshtanov, Dániel Marx, Marcin Pilipczuk, Michal Pilipczuk, and Saket Saurabh. *Parameterized Algorithms*. Springer, 2015. 1, 3
- [13] Reinhard Diestel. *Graph Theory*, volume 173 of *Graduate Texts in Mathematics*. Springer, 4th edition, 2010. 3, 7
- [14] Michael Dom, Daniel Lokshtanov, and Saket Saurabh. Kernelization lower bounds through colors and ids. *ACM Transactions on Algorithms*, 11(2):13, 2014. 2
- [15] Rodney G. Downey and Michael R. Fellows. *Fundamentals of Parameterized Complexity*. Texts in Computer Science. Springer, 2013. 1, 3
- [16] Andrew Drucker. New limits to classical and quantum instance compression. *SIAM Journal on Computing*, 44(5):1443–1479, 2015. 2
- [17] Pavel Dvořák and Dušan Knop. Parametrized complexity of length-bounded cuts and multi-cuts. In *Proc. of the 12th Annual Conference on Theory and Applications of Models of Computation (TAMC 2015)*, volume 9076 of *LNCS*, pages 441–452. Springer, 2015. 12
- [18] Jörg Flum and Martin Grohe. *Parameterized Complexity Theory*. Springer, 2006. 1, 3
- [19] Lance Fortnow and Rahul Santhanam. Infeasibility of instance compression and succinct PCPs for NP. *Journal of Computer and System Sciences*, 77(1):91–106, 2011. 2
- [20] Michael R. Garey, David S. Johnson, and Larry J. Stockmeyer. Some simplified NP-complete graph problems. *Theoretical Computer Science*, 1(3):237–267, 1976. 14
- [21] Petr A. Golovach and Dimitrios M. Thilikos. Paths of bounded length and their cuts: Pa-

- parameterized complexity and algorithms. *Discrete Optimization*, 8(1):72–86, 2011. 2, 12, 13, 22
- [22] Jiong Guo and Rolf Niedermeier. Invitation to data reduction and problem kernelization. *ACM SIGACT News*, 38(1):31–45, 2007. 1
- [23] Venkatesan Guruswami and Euiwoong Lee. Inapproximability of feedback vertex set for bounded length cycles. *Electronic Colloquium on Computational Complexity (ECCC)*, 21:6, 2014. 16
- [24] Lenwood Scott Heath. *Algorithms for Embedding Graphs in Books*. PhD thesis, University of North Carolina at Chapel Hill, USA, 1985. 20
- [25] Alon Itai, Yehoshua Perl, and Yossi Shiloach. The complexity of finding maximum disjoint paths with length constraints. *Networks*, 12(3):277–286, 1982. 12
- [26] Stefan Kratsch. Recent developments in kernelization: A survey. *Bulletin of the EATCS*, 113:58–97, 2014. 1, 4
- [27] Daniel Lokshantov, Dániel Marx, and Saket Saurabh. Lower bounds based on the exponential time hypothesis. *Bulletin of the EATCS*, 105:41–72, 2011. 1
- [28] Kavindra Malik, Ashok K. Mittal, and Santosh K. Gupta. The k most vital arcs in the shortest path problem. *Operations Research Letters*, 8(4):223–227, 1989. 12
- [29] Dániel Marx. Parameterized complexity and approximation algorithms. *The Computer Journal*, 51(1):60–78, 2008. 2
- [30] Karl Menger. Über reguläre Baumkurven. *Mathematische Annalen*, 96(1):572–582, 1927. 6
- [31] Bojan Mohar. Face covers and the genus problem for apex graphs. *Journal of Combinatorial Theory, Series B*, 82(1):102–117, 2001. 20
- [32] Rolf Niedermeier. *Invitation to Fixed-Parameter Algorithms*. Oxford University Press, 2006. 1, 3
- [33] Feng Pan and Aaron Schild. Interdiction problems on planar graphs. *Discrete Applied Mathematics*, 198:215–231, 2016. 12
- [34] Baruch Schieber, Amotz Bar-Noy, and Samir Khuller. The complexity of finding most vital arcs and nodes. Technical report, College Park, MD, USA, 1995. 20
- [35] Anneke A. Schoone, Hans L. Bodlaender, and Jan van Leeuwen. Diameter increase caused by edge deletion. *Journal of Graph Theory*, 11(3):409–427, 1987. 13, 14
- [36] Vijay V. Vazirani. *Approximation Algorithms*. Springer, 2001. 2
- [37] Douglas B. West. *Introduction to Graph Theory*. Prentice Hall, 2 edition, 2000. 3
- [38] David P. Williamson and David B. Shmoys. *The Design of Approximation Algorithms*. Cambridge University Press, 2011. 2
- [39] Ge Xia and Yong Zhang. On the small cycle transversal of planar graphs. *Theoretical Computer Science*, 412(29):3501–3509, 2011. 3, 16, 17
- [40] Ge Xia and Yong Zhang. Kernelization for cycle transversal problems. *Discrete Applied Mathematics*, 160(7-8):1224–1231, 2012. 16
- [41] Mihalis Yannakakis. Node- and edge-deletion NP-complete problems. In *Proc. of the 10th Annual Symposium on Theory of Computing (ACM 1978)*, pages 253–264. ACM, 1978. 16

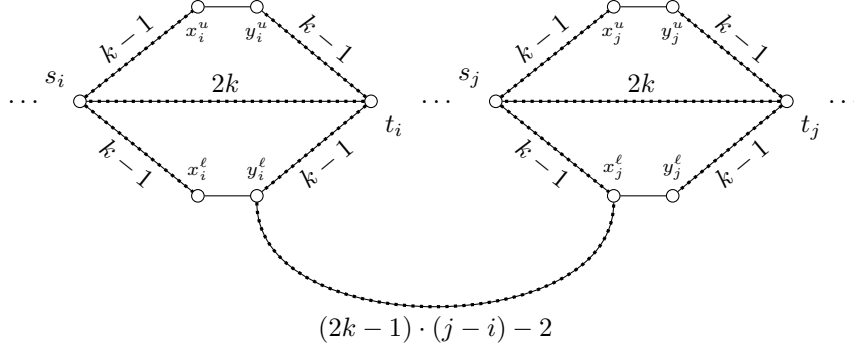


Figure 9: Illustration of the gadgets in the proof of Theorem 3. Here, exemplified for two vertices $i, j \in V$ with $\{i, j\} \in E$, and the edge is embedded on the second (lower) page in the two-page embedding of the input graph $G = (V, E)$.

A Proofs

A.1 Proof 1 (Theorem 3)

Definition 3. A page embedding of a graph G is a plane embedding of G where all vertices lie on the real line and every edge lies in the upper half $\mathcal{R} \times \mathcal{R}^+$.

Definition 4. A graph $G = (V, E)$ is k -page book embeddable if there is a partition E_1, \dots, E_k of the edge set E such that $G_i := (V, E_i)$ is page embeddable for all $i \in [k]$.

Proof. Our proof follows the same strategy as the proof due to Schieber et al. [34] for LBEC on general graphs, where Schieber et al. [34] reduce VERTEX COVER to LBEC. We reduce from 3-PLANAR VERTEX COVER, that is, VERTEX COVER on planar graphs with maximum degree three, which remains NP-complete [31]. Heath [24] proved that any planar graph of maximum vertex degree three allows a two-page embedding. Moreover, Heath showed that such an embedding can be computed in linear time in the number of vertices of the input graph. Recently, Bekos et al. [5] proved that any planar graph of maximum degree four allows a two-page embedding. We mainly copy the proof due to Schieber et al. [34] and, on the way, perform small changes on the gadgets and target parameters. We describe this in the following.

Let $\mathcal{I} = (G, k)$ be an instance of 3-PLANAR VERTEX COVER. Since we can assume to have a two-page embedding, the vertices are drawn along the real line and connected by non-crossing edges lying in the lower and upper half. Further, we assume that the vertices are labeled from 1 to n , in the order along the real line. We replace each vertex i by a gadget \mathbf{i} as follows. The gadget \mathbf{i} consists of two P_{2k} s, where P_{2k} denotes a simple path with $2k$ vertices, and one P_{2k+1} , all three merged together at their endpoints. We denote the left and right (merged) endpoint of gadget \mathbf{i} by s_i and t_i , respectively. One P_{2k} belongs to the upper half, the other to the lower half. The P_{2k+1} lies along the real line. The two middle vertices of each of the two P_{2k} we denote by $x_i^u, x_i^l, y_i^u, y_i^l$, where x is left of y , and u and l stand for “upper” and “lower”. We merge t_i with s_{i+1} for all $i \in [n-1]$. We set $s := s_1$ and $t := t_n$. Moreover, if two vertices $i < j$ are connected by an edge lying in the upper half, then we connect the vertex y_i^u with x_j^u via a path of length $(2k-1)(j-i)-2$ (analogously for edges in the lower half). Refer to Fig. 9 for an illustration of the construction. We denote by G' the obtained graph. Observe that G' remains planar. Except for the edges $\{x_i^u, y_i^u\}, \{x_i^l, y_i^l\}, i \in [n]$, there are no edges that are allowed to be deleted (see Schieber et al. [34]).

We set $k' := 2k$ and $\ell' = k \cdot (2k) + (n-k) \cdot (2k-1)$. Let $\mathcal{I}' := (G', s, t, k', \ell')$ be the resulting instance of PLANAR-LBEC(k', ℓ'), that is LBEC(k', ℓ') on planar graphs. We show that \mathcal{I} is a YES-instance if and only if \mathcal{I}' is a YES-instance.

“ \Rightarrow ”: Suppose that G admits a vertex cover of size at most k . Let $C \subseteq V(G)$ be such a vertex cover of size k . We claim that deleting the edges in the edge set $X := \{\{x_i^u, y_i^u\}, \{x_i^l, y_i^l\} \mid i \in C\}$ forms a solution to \mathcal{I}' .

We observe that any s - t path in $G' - X$ using only edges in the gadgets is of length at least ℓ' . To see this, consider a gadget \mathbf{i} with $i \in C$. Then the edges $\{x_i^u, y_i^u\}, \{x_i^\ell, y_i^\ell\} \in X$, and hence the only s_i - t_i path using only edges in the gadget \mathbf{i} is of length $2k$ (that is the P_{2k+1} used in the construction). If no edge in a gadget \mathbf{j} is deleted, then any shortest s_j - t_j path using only edges in the gadget \mathbf{j} is of length $2k - 1$ (those correspond to the P_{2k} s used in the construction). Since $|C| = k$, any s - t path in $G' - X$ using only edges in the gadgets is of length at least $k \cdot (2k) + (n - k) \cdot (2k - 1) = \ell'$.

We have to show that there is no shorter s - t path in $G' - X$ than any path using only edges in the gadgets. To this end, let $i, j \in V(G)$, $i < j$, be two adjacent vertices in G , that is, with $\{i, j\} \in E(G)$. Since C is a vertex cover, it follows that either $i \in C$ or $j \in C$. Let $i \in C$ and $j \notin C$ (the case with $j \in C$ and $i \notin C$ is symmetric). We consider the shortest path from s_i to t_j not going backwards, that is, not appearing in any gadget z with $z < i$ or $z > j$, and using the path connecting the gadgets of i and j . Let the path connecting the gadgets of i and j be a lower path, that is, the vertices y_i^ℓ and x_j^ℓ are connected by the path. Since the edges $\{x_i^\ell, y_i^\ell\}$ and $\{x_i^u, y_i^u\}$ are deleted, the shortest path from s_i to y_i^ℓ is of length $2k + (k - 1)$. Then we take the path of length $(2k - 1)(j - i) - 2$ to get to the gadget of j . Finally, we take the path from x_j^ℓ via edge $\{x_j^\ell, y_j^\ell\}$ to t_j of length $(k - 1) + 1 = k$. In total, the path is of length $4k + (2k - 1)(j - i) - 3$, and it is the shortest of its kind.

We compare this to the shortest path from s_i to t_j using only edges in the gadgets. The length of such a path is at most $2k(k) + (2k - 1)(j - i - k) + (2k - 1) = 2k(j - i) - (j - i - k) + (2k - 1)$ if $j - i \geq k$, and at most $2k(j - i) + (2k - 1)$ otherwise. Comparing the two lengths, we obtain for $j - i \geq k$

$$4k + (2k - 1)(j - i) - 3 - (2k(j - i) - (j - i - k) + (2k - 1)) = k - 2,$$

and for $j - i < k$

$$4k + (2k - 1)(j - i) - 3 - (2k(j - i) + (2k - 1)) = 2k - (j - i) - 2 > k - 2.$$

It follows that there is a path using only edges in the gadgets that is shorter than the shortest paths using at least one edge not appearing in the gadgets. Finally note that if both $i, j \in C$, then the difference of the path lengths is even bigger. Observe that using a path connecting gadget \mathbf{i} with $\mathbf{j} + 1$ (or $\mathbf{i} - 1$ with \mathbf{j}) to get from s_i to t_j is longer by at least $k - 1$ (or at least $k - 3$), following from an analogous argumentation as above. Hence, the shortest path connecting s with t passes through the gadgets and is of length at least ℓ' .

“ \Leftarrow ”: Suppose that G' allows $k' = 2k$ edge deletions such that any shortest s - t path is of length at least ℓ' . Our first observation is that in any solution to \mathcal{I} , either none or exactly two edges are deleted in any gadget. Suppose that there is an gadget with only one edge deleted. Then a shortest path through this gadget is of length $2k - 1$. Since $2k$ is the maximum increase of the passing length through a gadget, we get $(2k)(k - 2) + (2k - 1)(n - k + 2) < (2k) \cdot k + (2k - 1)(n - k) = \ell'$. Hence, in any gadget, either exactly two or no edge is deleted. Let $C \subseteq V(G)$ be the set of vertices such that both edges are deleted in the corresponding gadgets. We claim that C is a vertex cover of size k in G .

Suppose that there are two gadgets \mathbf{i} and \mathbf{j} not containing any deleted edge, that is, $\{i, j\} \cap C = \emptyset$, but $\{i, j\} \in E(G)$. Then the shortest s_i - t_j path using the path corresponding to edge $\{i, j\} \in E(G)$ is of length $2k + (2k - 1)(i - j) - 2$. The shortest s_i - t_j path through the gadgets only is of length at least $2k - 1 + (2k - 1)(i - j)$. Thus, the path using the path connecting the gadgets \mathbf{i} and \mathbf{j} is too short by exactly one, and hence, the shortest s - t path is of length smaller than ℓ' . This contradicts the fact that $\{\{x_i^u, y_i^u\}, \{x_i^\ell, y_i^\ell\} \mid i \in C\}$ forms a solution to \mathcal{I}' . It follows that for each edge $\{i, j\} \in E(G)$ we have $|C \cap \{i, j\}| > 0$. This is exactly the property of a vertex cover, and thus, C is a vertex cover in G of size k .

We have shown that the problem is NP-hard on planar, undirected graphs. Observe that we can direct all edges from “left to right”. The planarity still holds, and we obtain a directed acyclic graph. Since we have shown in the proof that “going backwards” is never optimal, the proof can be easily adapted. Thus, the problem remains NP-hard on planar directed acyclic graphs. \square

A.2 Proof 2 (Theorem 5)

Proof. We give a search tree algorithm branching over the possible edge deletions to prove that $\text{MDED}(k, \ell)$ is fixed-parameter tractable. The underlying crucial observation is that if some instance (G, k, ℓ) of $\text{MDED}(k, \ell)$ is a YES-instance, then there exists at least one pair of vertices $v, w \in V$ in the graph $G - X$ such that $\text{dist}_{G-X}(v, w) \geq \ell$, where X is a solution to (G, k, ℓ) . Hence, we want to check whether we can increase by at most k edge deletions the length of any shortest path between the chosen pair up to at least ℓ , where we delete an edge only if its deletion leaves the graph connected.

To this end, for each pair, we apply the branching algorithm provided by Golovach and Thilikos [21]: Find a shortest path and if its length is at most $\ell - 1$, then branch in all cases of deleting an edge on this path and decrease k by one. In each branch, we need to check whether the graph is still connected. This can be done in $O(n + m)$ time with a simple depth/breadth first search. Hence, in total we obtain a branching algorithm running in $O(n^2 \cdot (\ell - 1)^k(n + m))$ time. Thus, $\text{MDED}(k, \ell)$ is fixed-parameter tractable. \square

A.3 Proof 3 (Theorem 9)

Proof. We give a search tree algorithm branching over all possible edge deletions to prove that $\text{DSCT}(k, \ell)$ is fixed-parameter tractable. Let (G, k, ℓ) be an instance of $\text{DSCT}(k, \ell)$. To detect short cycles in G containing a vertex $v \in V(G)$, we construct a helping graph G_v as follows. Delete v (and all edges incident to v), and add v_{in} and v_{out} , and the arcs $\{(x, v_{\text{in}}) \mid (x, v) \in E(G)\}$, $\{(v_{\text{out}}, x) \mid (v, x) \in E(G)\}$ as well as the arc $(v_{\text{in}}, v_{\text{out}})$. Now to detect the shortest cycle in G containing v , compute a shortest $v_{\text{out}}-v_{\text{in}}$ path in G_v . If a cycle is too short, then we branch into all possible, at most ℓ different deletions of an arc of the cycle (beside arc $(v_{\text{in}}, v_{\text{out}})$).

The depth of the search tree is at most k , and thus we obtain an $O(\ell^k \cdot (n + m))$ -time algorithm since constructing G_v and finding a shortest path in unweighted graphs can be done in $O(n + m)$ time. \square